

# On the thermal stability of a static spherically symmetric black holes in Nash embedding framework

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**Abstract.** We study the deformation caused by the influence of extrinsic curvature on a vacuum spherically symmetric metric embedded in a five-dimensional bulk. In this sense, we investigate the produced black-holes and derive general characteristics such as their masses, horizons, singularities and thermal properties. As a test, we also study the bending of light near such black-holes analyzing the movement of a test particle and the modification caused by extrinsic curvature on its movement. Accordingly, using the asymptotically conformal flat condition for the extrinsic curvature, an analytical expansion of a set of  $n$ -scalar fields can be defined and we show that the corresponding black holes must be large and constrained in the range of allowed values  $-1/2 \leq n \leq 1.8$ . As a result, they are locally thermodynamically stable, but not globally preferred.

**Keywords:** modified gravity, astrophysical black holes

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## 1 Introduction

In a previous publication [1], we studied the influence of extrinsic curvature on the rotation curves of galaxies obtaining a good agreement with the observed velocity rotation curves of smooth hybrid alpha-HI measurements. In doing so, we used the smooth deformation concept [1–4] based on Nash theorem of embedding geometries [5]. This allowed us the possibility to develop a more general approach to embedded space-times. Instead of assuming the string inspired embedding of a three-dimensional hypersurface generating a four-dimensional embedded volume, we look at the conditions for the existence of the embedding of the space-time. The embedding of a manifold into another is a non-trivial problem and the resulting embedded geometry resulting from of an evolving three-surface must comply the Gauss-Codazzi-Ricci equations. For instance, many of the currently brane-world models fail to comply with those equations.

In the Randall-Sundrum brane-world model (RS) [6, 7], the space-time is embedded in a five dimensional anti-deSitter space AdS5. It is assumed that the space-time acts as a boundary for the higher-dimensional gravitational field. This condition has the effect that extrinsic curvature becomes an algebraic function of the energy-momentum tensor of matter confined to the four-dimensional embedded space-time. In more than five dimensions that boundary condition does not make sense because the extrinsic curvature acquire an internal index while the energy-momentum tensor does not have this degree of freedom. To obtain the so-called Israel condition it becomes necessary to consider the brane-world as a boundary separating two regions of the bulk, and find the difference between the expression calculated in both sides. The Israel condition only follows after the additional condition that the boundary brane-world acts as a mirror: Only in this case the tangent components cancel and the components depending on extrinsic curvature remain. This can be seen in the original paper by W. Israel [8], or in the appendix B of reference [2].

We should add that the Israel condition is not used in many other brane-world models, such as, for instance, the original paper by the Arkani-Hamed, G. Dvali and S. Dimopolous [9], sometimes referred as ADD model, and also the Dvali-Gabadadze-Porrati model [10]. Rather, it was shown that the Israel condition is not unique, so different conditions may lead to different physical results [11]. Other brane-world models have been developed with no need of particular junction conditions, see [1–4, 12–15] and references therein. Moreover,

we should also add that such condition is particular to five-dimensional bulk spaces and is not consistent with higher dimensional embeddings. In the case of a Schwarzschild solution [20, 21], for instance, the embedding would be compromised in RS scheme and evinces the necessity of a more general framework. Unless we are restricted to a class particular models the Israel condition does not apply because only the mirror symmetry on the brane-world is dropped.

In principle general relativity is exempt of the discussed Riemann curvature ambiguity problem because the ground state of the gravitational field, the Minkowski space-time, is well defined by the existence of Poincaré symmetry (in special the translations). Such situation between particle physics and Einstein’s gravity was shaken by the observations of a small cosmological constant, then the emergence of the cosmological constant problem. This leads us back to the necessity of the embedding of space-times as the only known solution of the Riemann curvature ambiguity.

The general solution of the embedding problem (mainly on Gauss-Codazzi-Ricci equations) is shown in the paper by J. Nash on differentiable embedding [5], and generalized by Greene [22] including positive and negative signatures. The application of Nash theorem to gravitation is a landmark of our paper, providing a new tool to the gravitational perturbation issue. The embedding mechanics presented here is a legitimate condition in all other more general settings.

The aim of our present work is to focus on the classical thermal stability of black holes embedded in five-dimensional bulk space-time. Recently, the study of the high dimensional space-times and the prediction of a new black objects (black holes, black strings, black rings, and so on) have been the focus of very active research [16–19]. Specifically, it is worth noting that a Schwarzschild embedding was a problem discussed a long ago [20, 21, 26–28] and it is known that this geometry is completely embedded in six-dimensions [29]. In particular, the effects of a five dimensional bulk space-time endowed with an embedded spherically symmetric metric is still an open research arena since we do not have a closed analytical solution [30–33]. Notwithstanding, it may lead to new physical consequences in a search of a full description and understanding of astrophysical objects, mainly on stellar dynamics and gravitational collapse. Moreover, in this work we explore the consequences of such embedding by using the asymptotic condition for extrinsic curvature in order to attenuate such restrictions.

The paper is organized as follows. In section 2, we briefly present a summary of the theoretical background of this paper restricted to the five dimensional space-time. In section 3, we study a spherically symmetric metric embedded in five dimensions. Moreover, in section 4, we analyze the constrains on the solution and the resulting horizons, which will be particular important to the thermodynamic analysis as presented in section 5. Finally, in the conclusion section, the final remarks and future prospects are presented.

## 2 The induced four dimensional equations

The present paper is not M-theory/Strings related. As well-known, the Randall-Sundrum (RS) model [6, 7] is a brane-world theory originated from M-theory in connection with the derivation of the Horava-Witten heterotic  $E8 \times E8$  string theory in the space  $AdS5 \times S^5$ . In that framework, one obtains a compactification of one extra dimension on the orbifold  $S^1/Z_2$  by using the  $Z_2$  symmetry on the circle  $S^1$ . The presence of the five dimensional anti-deSitter  $AdS_5$  space is mainly motivated by the prospects of the AdS/CFT correspondence between

the superconformal Yang-Mills theory in four dimensions and the anti-deSitter gravity in five dimensions.

In a different approach, our proposal is initially based on the possibility to explore embedded space-times in a search of obtaining a more general theory based on embeddings (the brane-world models, of course, can be an example). Instead of assuming the string inspired embedding of a three-dimensional hypersurface generating a four-dimensional embedded volume, we look at the conditions for the embedding of the space-time itself.

Nash's original theorem used a flat D-dimensional Euclidean space but this was soon generalized to any Riemannian manifold, including those with non-positive signatures [22]. For simplicity, we refer the term "Nash theorem" as valid also for pseudo-Riemannian manifolds. Another important aspect is that the seminal work on perturbation of geometry was proposed posthumously in 1926 by Campbell [23] using analytic embedding functions and later extended to non-positive signatures [24]. Although the Nash theorem could also be generalized to include perturbations on arbitrary directions in the bulk, it would make its interpretations more difficult, so that we retain Nash's choice of independent orthogonal perturbations. It should be noted that the smoothness of the embedding is a primary concern of Nash theorem. In this respect, the natural choice for the bulk is that its metric satisfy the Einstein-Hilbert principle. Indeed, that principle represents a statement on the smoothness of the embedding space (the variation of the Ricci scalar is the minimum possible). Admitting that the perturbations are smooth (differentiable), then the embedded geometry will be also differentiable.

As it happens, Nash showed that any embedded unperturbed metric  $g_{\mu\nu}$  can be generated by a continuous sequence of small metric perturbations of a given geometry with a metric of the embedded space-time

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta y^a k_{\mu\nu a} + \delta y^a \delta y^b g^{\rho\sigma} k_{\mu\rho a} k_{\nu\sigma b} + \dots \quad (2.1)$$

or, equivalently

$$k_{\mu\nu} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial y}, \quad (2.2)$$

where  $k_{\mu\nu}$  is the initially unperturbed extrinsic curvature and  $y$  is the coordinate related to extra dimensions. Since Nash's smooth deformations are applied to the embedding process, the coordinate  $y$ , usually noticed in rigid embedded models, e.g [6, 7], can be omitted in the process for perturbing an element line (depending on what one wants to do there are many different ways to embed a manifold into another classified as local, global, isometric, conformal or more generally defined by a collineation, rigid, deformable, analytic or differentiable embedding). Differently from RS models, using Nash embedding framework, we can develop a model that considers the extrinsic curvature as a dynamical (physical) quantity [1–4, 12–15]. Also, the paper should not be confused with a paper on quantization deformation, since our arguments are all classical.

To start with, we consider an example with a Riemannian manifold  $\bar{V}_4$  with metric  $\bar{g}_{\mu\nu}$ , and its local isometric embedding in a five-dimensional Riemannian manifold  $V_5$ , given by a differentiable and regular map  $\bar{X} : \bar{V}_4 \rightarrow V_5$  satisfying the embedding equations

$$\bar{X}^A_{,\mu} \bar{X}^B_{,\nu} \mathcal{G}_{AB} = g_{\mu\nu}, \quad \bar{X}^A_{,\mu} \bar{\eta}^B \mathcal{G}_{AB} = 0, \quad \bar{\eta}^A \bar{\eta}^B \mathcal{G}_{AB} = 1, \quad A, B = 1..D \quad (2.3)$$

where we have denoted by  $\mathcal{G}_{AB}$  the metric components of  $V_5$  in arbitrary coordinates, and where  $\bar{\eta}$  denotes the unit vector field orthogonal to  $\bar{V}_4$ . Concerning notation, capital Latin

indices run from 1 to 5. Small case Latin indices refer to the only one extra dimension considered. All Greek indices refer to the embedded space-time counting from 1 to 4. In general we have  $D = n + 1$  with the index  $D$  representing the bulk space dimension and the index  $n$  represents the embedded space dimension.

The extrinsic curvature of  $\bar{V}_n$  is by definition the projection of the variation of  $\eta$  on the tangent plane :

$$\bar{k}_{\mu\nu} = -\bar{X}^A{}_{,\mu}\bar{\eta}^B{}_{,\nu}\mathcal{G}_{AB} = \bar{X}^A{}_{,\mu\nu}\bar{\eta}^B\mathcal{G}_{AB} \quad (2.4)$$

The integration of the system of equations Eq.(2.3) gives the required embedding map  $\bar{X}$ . Moreover, one can construct the one-parameter group of diffeomorphisms defined by the map  $h_y(p) : V_D \rightarrow V_D$ , describing a continuous curve  $\alpha(y) = h_y(p)$ , passing through the point  $p \in \bar{V}_n$ , with unit normal vector  $\alpha'(p) = \eta(p)$ . Thus, the group is characterized by the composition  $h_y \circ h_{\pm y'}(p) \stackrel{def}{=} h_{y \pm y'}(p)$ ,  $h_0(p) \stackrel{def}{=} p$ . Accordingly, applying this diffeomorphism to all points of a small neighborhood of  $p$ , we obtain a congruence of curves (or orbits) orthogonal to  $\bar{V}_n$ . It does not matter if the parameter  $y$  is time-like or not, nor if it is positive or negative.

If one defines a geometric object  $\bar{\omega}$  in  $\bar{V}_n$ , its Lie transport along the flow for a small distance  $\delta y$  is given by  $\Omega = \bar{\Omega} + \delta y \mathcal{L}_\eta \bar{\Omega}$ , where  $\mathcal{L}_\eta$  denotes the Lie derivative with respect to  $\eta$ . In particular, the Lie transport of the Gaussian frame  $\{\bar{X}_\mu^A, \bar{\eta}_a^A\}$ , defined on  $\bar{V}_n$  gives

$$Z^A{}_{,\mu} = X^A{}_{,\mu} + \delta y \mathcal{L}_\eta X^A{}_{,\mu} = X^A{}_{,\mu} + \delta y \eta^A{}_{,\mu} \quad (2.5)$$

$$\eta^A = \bar{\eta}^A + \delta y [\bar{\eta}, \eta]^A = \bar{\eta}^A \quad (2.6)$$

However, from Eq.(2.4) we note that in general  $\eta_{,\mu} \neq \bar{\eta}_{,\mu}$ .

In order to describe another manifold, the set of coordinates  $Z^A$  need to satisfy the embedding equations similar to Eq.(2.3) as

$$Z^A{}_{,\mu} Z^B{}_{,\nu} \mathcal{G}_{AB} = g_{\mu\nu}, \quad Z^A{}_{,\mu} \eta^B \mathcal{G}_{AB} = 0, \quad \eta^A \eta^B \mathcal{G}_{AB} = 1 \quad (2.7)$$

Replacing Eq.(2.5) and Eq.(2.6) in Eq.(2.7) and using the definition from Eq.(2.4), we obtain the metric and extrinsic curvature of the new manifold

$$g_{\mu\nu} = \bar{g}_{\mu\nu} - 2y \bar{k}_{\mu\nu} + y^2 \bar{g}^{\rho\sigma} \bar{k}_{\mu\rho} \bar{k}_{\nu\sigma} \quad (2.8)$$

$$k_{\mu\nu} = \bar{k}_{\mu\nu} - 2y \bar{g}^{\rho\sigma} \bar{k}_{\mu\rho} \bar{k}_{\nu\sigma} \quad (2.9)$$

Taking the derivative of Eq.(2.8) with respect to  $y$  we obtain Nash's deformation condition as shown in Eq.(2.2). Moreover, the integrability conditions for these equations are given by the non-trivial components of the Riemann tensor of the embedding space<sup>1</sup>.

<sup>1</sup>To avoid confusion with the four dimensional Riemann tensor  $R_{\alpha\beta\gamma\delta}$ , expressed in the frame  $\{Z_\mu^A, \eta^A\}$  as

$${}^5\mathcal{R}_{ABCD} Z^A{}_{,\alpha} Z^B{}_{,\beta} Z^C{}_{,\gamma} Z^D{}_{,\delta} = R_{\alpha\beta\gamma\delta} + (k_{\alpha\gamma} k_{\beta\delta} - k_{\alpha\delta} k_{\beta\gamma}) \quad (2.10)$$

$${}^5\mathcal{R}_{ABCD} Z^A{}_{,\alpha} Z^B{}_{,\beta} Z^C{}_{,\gamma} \eta^D = k_{\alpha[\beta;\gamma]} \quad (2.11)$$

These are the mentioned Gauss-Codazzi equations (the third equation -the Ricci equation- does not appear in the case of just one extra dimension). The first of these equation (Gauss) shows that the Riemann curvature of the embedding space acts as a reference for the Riemann curvature of the embedded space-time. The second equation (Codazzi) complements this interpretation, stating that projection of the Riemann tensor of the embedding space along the normal direction is given by the tangent variation of the extrinsic curvature. Notice from Eq.(2.10) that the local shape of an embedded Riemannian manifold is determined not only by its Riemann tensor but also by its extrinsic curvature, completing the proof of the Schläfli embedding conjecture by use of Nash's deformation condition in Eq.(2.2). This guarantees to reconstruct the five-dimensional geometry and understand its properties from the dynamics, in this case, of the four-dimensional embedded space-time.

As in Kaluza-Klein and in the brane-world theories, the embedding space  $V_5$  has a metric geometry defined by the higher-dimensional Einstein's equations for the bulk in arbitrary coordinates

$$\mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\mathcal{G}_{AB} = \alpha_* T_{AB}^*, \quad (2.12)$$

where the metric of the bulk is denoted by  $\mathcal{G}_{AB}$  and we have dispensed the bulk with a cosmological constant, since for the present application in astrophysical scale the induced four-dimensional cosmological constant has a very small value and can be neglected. The energy-momentum tensor for the bulk of the known matter and gauge fields is denoted by  $T_{AB}^*$ . The constant  $\alpha_*$  determines, in this case, the five-dimensional energy scale.

In what concerns the confinement, the four-dimensionality of the space-time is an experimentally established fact, associated with the Poincaré invariance of Maxwell's equations and their dualities, later extended to all gauge fields. Therefore, all matter which interacts with these gauge fields must be defined in the four-dimensional space-times. On the other hand, in spite of all efforts made so far, the gravitational interaction has failed to fit into a similar gauge scheme, so that the gravitational field does not necessarily have the same four-dimensional limitations and accesses the extra dimensions in accordance with Eq.(2.1), regardless the location of its sources.

We assume that the four-dimensionality of gauge fields and ordinary matter applies to all perturbed space-times, so that it corresponds to a confinement condition. In order to recover Einstein's gravity by reversing the embedding, the confinement of ordinary matter and gauge fields implies that the tangent components of  $\alpha_* T_{AB}^*$  in Eq.(2.12) must coincide with  $(8\pi G T_{\mu\nu})$ , where  $T_{\mu\nu}$  is the induced four-dimensional energy-momentum tensor of the confined sources. As it may have been already noted, we are essentially reproducing the brane-world program, with the difference that it is a general approach and has nothing to do with branes in string/M theory. Instead, all that we use here is Nash theorem together with the four-dimensionality of gauge fields, the Einstein-Hilbert principle for the bulk and a D-dimensional energy scale  $\alpha_*$ . The confinement is set simply as  $\alpha_* T^* = 8\pi G T_{\mu\nu}$ , where  $T_{\mu\nu}$  represents the energy momentum tensor of the confined matter, and  $T_{\mu 5}^* = T_{55}^* = 0$ . This can be understood in more general terms as a consequence of the duality operations of the Yang-Mills equations. It follows that  $D \wedge F^*$  is a three-form that must be isomorphic to the current 1-form  $j^*$ . This condition depends only on the four-dimensionality of the space-time [34, 35] consistent with a plethora of experimental facts. Even that any gauge theory can be mathematically constructed in a higher dimensional space, we adopt the confinement as a condition, since the four-dimensionality of space-time will suffice in our case based on experimentally high-energy tests suggest [25].

With the above remarks, we may re-write the components of Eq.(2.12), using Eq.(2.2) and Eq.(2.7) together with the previous confinement conditions [2] obtaining

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - Q_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (2.13)$$

$$k_{\mu;\rho}^\rho - h_{,\mu} = 0, \quad (2.14)$$

where the term  $Q_{\mu\nu}$  results from the expression of  $\mathcal{R}_{AB}$  in Eq.(2.12), involving the orthogonal and mixed components of the Christoffel symbols

$$Q_{\mu\nu} = g^{\rho\sigma} k_{\mu\rho} k_{\nu\sigma} - k_{\mu\nu} h - \frac{1}{2} (K^2 - h^2) g_{\mu\nu}, \quad (2.15)$$

where  $h^2 = g^{\mu\nu} k_{\mu\nu}$  is the (squared) mean curvature and  $K^2 = k^{\mu\nu} k_{\mu\nu}$  is the (squared) Gauss curvature. This quantity is therefore entirely geometrical and it is conserved in the sense of

$$Q^{\mu\nu}{}_{;\nu} = 0 . \quad (2.16)$$

Therefore we may derive observable effects associated with the extrinsic curvature capable to be seen by four-dimensional observers in space-times. A detailed derivation of these equations can be found in [2–4] and references therein. Hereafter, we use a system of unit such that  $c = G = 1$ .

### 3 Induced spherically symmetric vacuum solution

In the present study, we focus our work on obtaining an exterior vacuum spherical solution, i.e,  $T_{\mu\nu} = 0$ . To this end, we start with the general static spherically symmetric induced metric that can be described by the line element

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 , \quad (3.1)$$

where we denote the functions  $A(r) = A$  and  $B(r) = B$ . Thus, one can obtain the following components for the Ricci tensor:

$$\begin{aligned} R_{rr} &= \frac{B''}{2B} - \frac{1}{4} \frac{B'}{B} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{A'}{A} \\ R_{\theta\theta} &= -1 + \frac{r}{2A} \left( -\frac{A'}{A} + \frac{B'}{B} \right) + \frac{1}{A} \\ R_{\phi\phi} &= \sin^2 \theta R_{\theta\theta} \\ R_{tt} &= -\frac{B''}{2A} + \frac{1}{4} \frac{B'}{A} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{B'}{A} \end{aligned}$$

where we have  $\frac{dA}{dr} = A'$  e  $\frac{dB}{dr} = B'$ .

From Eq.(2.13), the gravitational-tensor vacuum equations (with  $T_{\mu\nu} = 0$ ) can be written in alternative form as

$$R_{\mu\nu} + \frac{1}{2} Q g_{\mu\nu} = Q_{\mu\nu} \quad (3.2)$$

where we use the contraction  $Q = g^{\mu\nu} Q_{\mu\nu}$ .

The general solution of Codazzi equations Eq.(2.14) is given by

$$k_{\mu\nu} = f_\mu g_{\mu\nu} \quad (\text{no sum on } \mu) , \quad (3.3)$$

Taking the former equation and the definition of  $Q_{\mu\nu}$ , one can write

$$Q_{\mu\nu} = f_\mu^2 g_{\mu\nu} - \sum_\alpha f_\alpha f_\mu g_{\mu\nu} - \frac{1}{2} \left( \sum_\alpha f_\alpha^2 - \left( \sum_\alpha f_\alpha \right)^2 \right) g_{\mu\nu} ,$$

where

$$U_\mu = f_\mu^2 - \left( \sum_\alpha f_\alpha \right) f_\mu - \frac{1}{2} \left( \sum_\alpha f_\alpha^2 - \left( \sum_\alpha f_\alpha \right)^2 \right) \delta_\mu^\mu .$$

Consequently, we can write  $Q_{\mu\nu}$  in terms of  $f_\mu$  as

$$Q_{\mu\nu} = U_\mu g_{\mu\nu} \quad (\text{no sum on } \mu). \quad (3.4)$$

Since  $Q_{\mu\nu}$  is a conserved quantity, we can find four equations from  $\sum_\nu g^{\mu\nu} U_{\mu;\nu} = 0$  that can be reduced to only two equations:

$$\begin{cases} (f_1 + f_2)(f_3 - f_4) = 0 ; \\ (f_3 + f_4)(f_1 - f_2) = 0 . \end{cases}$$

and result in the condition

$$f_1 f_3 = f_2 f_4 . \quad (3.5)$$

A straightforward consequence of the homogeneity of Eq.(2.14) and the condition Eq.(3.5), the individual arbitrariness of the functions  $f_\mu$  can be reduced to a unique arbitrary function  $\alpha$  that depends on the radial coordinate. Hence, the equation (3.3) turns out to be

$$k_{\mu\nu} = \alpha(r) g_{\mu\nu} , \quad (3.6)$$

with  $\alpha(r) = \alpha$ .

Thus, we can determine the extrinsic quantities

$$Q_{\mu\nu} = 3\alpha^2 g_{\mu\nu} , \quad (3.7)$$

$$Q = Tr(Q_{\mu\nu}) = 12\alpha^2 , \quad (3.8)$$

and substituting in the gravitational-tensor equation Eq.(2.13), one can obtain the following  $rr$  and  $tt$  components:

$$\frac{B''}{2B} - \frac{1}{4} \frac{B'}{B} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{A'}{A} = 9\alpha^2(r)A , \quad (3.9)$$

$$-\frac{B''}{2A} + \frac{1}{4} \frac{B'}{A} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{1}{r} \frac{B'}{A} = -9\alpha^2(r)B . \quad (3.10)$$

From these equations, we have

$$\frac{A'}{A} = -\frac{B'}{B} .$$

thus,  $AB = \text{constant}$ .

In the same way as the calculation of the very known Schwarzschild vacuum solution, we impose the contour

$$\lim_{r \rightarrow \infty} A(r) = \lim_{r \rightarrow \infty} B(r) = 1 ,$$

in order to approach the metric tensor to Minkowski tensor as  $r \rightarrow \infty$ , we have

$$A(r) = \frac{1}{B(r)} .$$

Furthermore, using the  $(\theta, \theta)$  component, one can obtain

$$B(r) = 1 + \frac{k}{r} + \frac{9}{r} \int \alpha^2(r) r^2 dr , \quad (3.11)$$



and

$$A(r) = [B(r)]^{-1} = \left[ 1 + \frac{k}{r} + \frac{9}{r} \int \alpha^2(r) r^2 dr \right]^{-1}. \quad (3.12)$$

where  $k$  is a constant.

Since spherically symmetric geometry is very constrained in a five-dimensional embedding [29]. Based on the behavior of extrinsic curvature at infinity, we set the asymptotically conformal flat condition on extrinsic curvature as

$$\lim_{r \rightarrow \infty} k_{\mu\nu} = \lim_{r \rightarrow \infty} \alpha(r) \lim_{r \rightarrow \infty} g_{\mu\nu}. \quad (3.13)$$

It is important to stress that the Nash theorem *per se* does not provide a dynamical set of equations for extrinsic curvature but shows how to relate extrinsic curvature to the metric through a smooth process of deformation making the local embedding regular and completely differentiable. As  $\lim_{r \rightarrow \infty} g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ , where  $\eta_{\mu\nu}$  is Minkowskian  $M_4$  metric, the extrinsic curvature vanishes as it tends to infinity, so the function  $\alpha(r)$  must comply with this condition. Thus, we can infer that the function  $\alpha(r)$  must be analytical at infinity and to attend this constraint, we choose the simplest option

$$\alpha(r) = \sum_{n=i}^s \frac{\sqrt{-\alpha_0}}{\gamma^* r^n}, \quad (3.14)$$

where the sum is upon all scalar potentials and the indices  $i$  and  $s$  are real numbers. The index  $n$  may represent a set of scalar fields. The parameter  $\alpha_0$  has cosmological magnitude with the same units as the Hubble constant and its modulus is estimated as 0.677 [1] that will be used hereon. In order to keep the right dimension to Eq.(3.14), we have introduced a unitary parameter  $\gamma^*$  that has the inverse unit of Hubble constant [1] and defines the cosmological horizon in Eq.(3.17). From the geometrical point of view, Eq.(3.14) will not produce umbilical points on this spherically surface leading to a local additional bending in space-time.

Using Eqs.(3.11) and (3.14), one can obtain an explicit form of the coefficient  $B(r)$  given by

$$B(r) = 1 - \frac{K(9\alpha_0^2 + 1)}{r} - \sum_{n=i}^s \frac{9\alpha_0}{\gamma^*(2n-3)} r^{2(1-n)}. \quad (3.15)$$

In terms on the correspondence principle with Einstein gravity, we set  $K(9\alpha_0 + 1) = -4M$ , which remains valid even in the limit when  $\alpha \rightarrow 0$  in order to obtain the asymptotically flat solution. In addition, the related potential is given by

$$\Phi(r) = -1 + \frac{2M}{r} + \sum_{n=i}^s \frac{9\alpha_0}{\gamma^*(3-2n)} r^{2(1-n)}, \quad (3.16)$$

evinced how the initially spherical symmetric geometry is modified by the influence of the extrinsic terms. Thus, we can write the line element in Eq.(3.1) modified by extrinsic curvature as

$$ds^2 = \Phi(r) dt^2 - \frac{1}{\Phi(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (3.17)$$

for all  $n \neq \frac{3}{2}$ . Hence, one can obtain physical singularities with the calculation of the induced four-dimensional Kretschmann scalar  $\tilde{K}$  and find

$$\tilde{K} = \frac{g(r, \alpha_0/\gamma^*, M, n)}{r^6 (3-2n)^2 ((r-2M)(3-2n)r^{2n} + 9(\alpha_0/\gamma^*)^2 r^3)^6},$$

r	$\alpha_0$	n	Mass	$g(r, \alpha_0, M, n)$	$\tilde{K}$
0	0	$0 < n \leq 6$	any	0	undefined
0	0	0	any	$> 0$	diverges
0	0.677	$0 \leq n \leq 6$	any	$> 0$	diverges
$> 0$	0	any	0	0	0
$> 0$	0	any	$> 0$	0	$> 0$
$> 0$	0.677	$-10 \leq n \leq 10$	$\geq 0$	$> 0$	$> 0$

**Table 1.** A summary of relevant values for the term  $g(r, \alpha_0, M, n)$  in determining different behaviors of Kretschmann scalar  $\tilde{K}$ .

which preserves at first an intrinsic, physical and irremovable singularity at  $r = 0$  and  $n = \frac{3}{2}$ .

Another situations can be found when considering the term  $g(r, \alpha_0, M, n)$ . This term represents a very large polynomial function with 384 terms that were omitted here in its explicit form. Bearing in mind that the value of the extrinsic parameter  $\alpha_0$  has magnitude 0.677 or can be zero (if considering a vanishing extrinsic curvature) and analyzing the behavior of  $g(r, \alpha_0, M, n)$ , one can summarize in the following table other singularities, a vanishing and positive  $\tilde{K}$ .

As we can see, from the second to the fourth row in table (1) for any different values of  $n$ , the term  $g(r, \alpha_0, M, n)$  diverges leading to an undefined  $\tilde{K}$ . The fifth and sixth rows show an expected result since Riemannian curvature takes over when we have a vanishing extrinsic curvature. Interestingly, the last row shows the influence of the distortion cause by extrinsic curvature since even with a (non-forming) black hole with zero-mass [36], local tilde effects can be found once we do not have as a result the Minkowskian flat space-time. The range  $-10 \leq n \leq 10$  was found once a larger range induces a very fast growing and decaying of the exponents but not observed in solar scales. In the section 4, we will study the possibility to get a more constrained range for those values of  $n$ .

For the sake of completeness, we test the bending of light near the horizons in this geometry warped by Nash's embedding. Using the line element for spherical symmetry in Eq.(3.1), one can study the geodesic motion given as the solution of the following equations

$$\begin{aligned}
\frac{d}{d\tau}(-At) &= 0, \\
\frac{d}{d\tau}(r^2\dot{\theta}) - r^2 \sin \theta \cos \theta \dot{\phi}^2 &= 0, \\
\frac{d}{d\tau}(r^2 \sin^2 \theta \dot{\phi}) &= 0 \\
-At^2 + A^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 &= 0,
\end{aligned} \tag{3.18}$$

where  $\tau$  is the proper time and  $\frac{dx}{d\tau} \equiv \dot{x}$ . In addition, considering that the movement of a test particle takes place on a plane  $\theta = \pi/2$ , we perform a change of variables  $U = r^{-1}$  and find the following equation of trajectory

$$\frac{d^2U}{d\phi^2} + AU + \left(\frac{dA}{dU}\right) \left(\frac{U^2}{2}\right) = 0, \tag{3.19}$$

which yields

$$\frac{d^2 U}{d\phi^2} + U = 3MU + \sum_{n=i}^s \frac{9n(\alpha_0/\gamma^*)U^{2n-1}}{(3-2n)}. \quad (3.20)$$

Moreover, one can obtain the deflection angle as given by

$$\Delta\phi = \sum_{n=i}^s \int_b^\infty \frac{2dr}{r \left[ \left( \frac{r}{b} \right)^2 - A \right]^{1/2}} - \pi, \quad (3.21)$$

where the parameter  $b$  is the minimal distance from the source. Therefore,

$$\Delta\phi = \frac{4M}{b} - \sum_{n=i}^s \int_b^\infty \frac{9\alpha_0/\gamma^* r^{1-2n} dr}{(3-2n) \left[ \left( \frac{r}{b} \right)^2 - 1 \right]^{3/2}}. \quad (3.22)$$

We note that

$$\int \frac{r^{1-2n} dr}{\left[ \left( \frac{r}{b} \right)^2 - 1 \right]^{3/2}} = - \left( \frac{r^{2-2n} {}_2F_1(3/2, 1-n, 2-n, r^2/b^2)}{2(1-n)} \right), \quad (3.23)$$

where  ${}_2F_1$  is the gaussian hypergeometric function. Thus, one can find a generic expression for the shift  $\Delta\phi$  as

$$\Delta\phi = \Delta\phi_{GR} + \Delta\phi_d, \quad (3.24)$$

where  $\Delta\phi_{GR} = \frac{4M}{b}$  is the classical result provided from general relativity, and the second term  $\Delta\phi_d$  which is given by

$$\Delta\phi_d = \frac{9\alpha}{2\gamma^*} \sum_{n=i}^s \frac{r^{2(1-n)}}{(2n^2 - 5n + 3)} {}_2F_1(3/2, 1-n, 2-n, r^2/b^2), \quad (3.25)$$

which measures how strong is the deviation that depends on how intense is the deformation on the local space-time. For the sake of completeness, if we neglect  $\alpha_0$ , one obtains the same result obtained from general relativity.

#### 4 Constraints on the allowed values of the $n$ -scalar fields

In order to study the possible horizons by the element line in Eq.(3.17), we must constrain the values of the set of  $n$ -scalar fields. We start with analyzing the behavior of the gravitational potential on solar scales. In general, black holes are objects placed in galactic scales ruled essentially by newtonian potentials ( $\sim \frac{1}{r}$ ) and its smooth deviations in decaying ( $\sim \frac{1}{r^2}, \sim \frac{1}{r^3}$ ) and growing not large as  $\sim r^2$ . The increasing of gravitational potentials of order of  $\sim r^3$  possibly can lead to a topological deformation of the space-time [37] or quadratically, and so on, should be quite worrisome and inconsistent with the solar system scale that turns the possible values of  $n$  dramatically constrained. Bearing this in mind, we analyze the allowed values of  $n$ , since we already have a first insight from the analysis of Kretschmann scalar in table (1).

The general expression for the horizons of Eq.(3.17) can be found when we set  $g_{tt} = 0$ , so one can find

$$\sum_{n=i}^s \frac{9\alpha_0}{3-2n} r^{3-2n} + r - 2M = 0, \quad (4.1)$$

which is a polynomial equation of order  $(3 - 2n)$  that gives a class of horizons to be defined for all allowed values of  $n$ . Moreover, the corresponding mass is given by

$$M = \frac{1}{2} \left[ \sum_{n=i}^s \frac{9\alpha_0}{3-2n} r^{2n-3} + r \right]. \quad (4.2)$$

In order to avoid overloaded notation, the unitary parameter  $\gamma^*$  is dividing the parameter  $\alpha_0$  hereon. Hence, we obtain interesting cases for the allowed values of  $n$  since the value  $n = 3/2$  is discarded due to the fact that it diverges Eq.(3.15).

For instance, the Schwarzschild horizon can be straightforwardly obtained when we set  $\alpha_0 = 0$ , which means that extrinsic curvature vanishes and the space-time turns out to be asymptotically Minkowskian. For  $n = 0$ , one can recover the Schwarzschild-de Sitter-like solution and  $n = 2$  mimics Reissner-Nördstrom solution. It is important to say that the Reissner-Nördstrom-like solution the term  $\alpha_0^2$  is interpreted as a tidal charge but not originated from a electromagnetic field, since it is a vacuum solution. Interestingly, in this particular case, differently from RS models, our tidal charge  $\alpha_0$  is originated from the extrinsic curvature and has a new physics associated to it, since it has a cosmological magnitude. The solution when  $n = 1$  alone is a modification of the Schwarzschild solution by a term  $9\alpha_0$ .

For  $n \geq 3$ , the solutions produce a very low newtonian potentials ( $\sim \frac{1}{r^4}$  and on). On the other hand, for  $n \leq -1$  the potentials grows very fast. Both limits are not verified in solar system scales. Accordingly, we can set that the parameter  $n$  in summation must vary such as  $-1/2 < n < 3$  hereon.

An interesting case can be found when we set  $n = 1/2$  where we have found the equation

$$r - 2M - \alpha_0 r^2 = 0,$$

with cosmological horizon  $r_c = \frac{1+\sqrt{1-8\alpha_0 M}}{2\alpha_0}$  and black hole horizon  $r_{bh} = \frac{1-\sqrt{1-8\alpha_0 M}}{2\alpha_0}$ . Moreover, they exist when  $1-8\alpha_0 M > 0$  for  $M = \frac{1}{8\alpha_0}$  and both horizons coalesce at  $r_c = r_{bh} = \frac{1}{2\alpha_0}$ . Interestingly, taking eq.(3.15), one can set  $n = \frac{3-\beta}{2}$ , where  $\beta$  is a parameter, and find the same power law found in [38], which was used for an alternative model to explanation of the dark matter problem. Another curious fact from the richness of this model is that when  $n = 5/2$ , which mimics an asymptotically metric function for a Bardeen black hole [39].

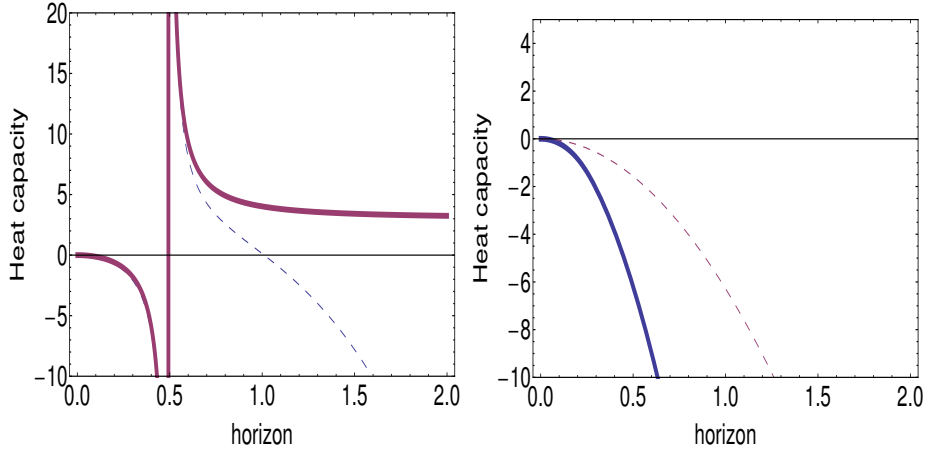
## 5 Thermodynamical stability

In order to study the classical stability of this model, we focus on the determination of heat capacity and the resulting free energy. To this end, we calculate the related Hawking temperature, which depends on how curved is the space near a black hole. Therefore, one can find the local surface gravity as

$$\kappa = -\frac{1}{2} \frac{dg_{44}}{dr} \Big|_{r=r_h(\text{horizon})} = \frac{M}{r_h} 9\alpha_0^2 \frac{(1-n)}{(3-2n)} r_h^{1-2n}.$$

Consequently, the Hawking temperature is given by

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_h^2} [r_h + 9\alpha_0^2 r_h^{3-2n}] , \quad (5.1)$$



**Figure 1.** The heat capacity is plotted as a function of the horizon with the allowed values of  $n$ . In the left panel is shown the behavior for the range in the allowed values  $-1/2 \leq n \leq 1.8$  (solid line) and for  $-1/2 \leq n \leq 3$  (dashed line). In the right panel, we compare the case  $-1/2 \leq n \leq 3$  (solid line) to Schwarzschild solution (dashed line).

where we denote the event horizon  $r_h$ . For the sake of completeness, we point out that when  $\alpha_0 \rightarrow 0$  we recover Schwarzschild with event horizon  $r_h = 2M$  and Hawking temperature  $T_h = \frac{1}{8\pi M}$ .

In addition, we determine the entropy defined by Bekenstein-Hawking formulae as

$$S = \int_0^{r_h} \frac{1}{T} \frac{\partial M}{\partial r} dr ,$$

and using the corresponding mass in Eq.(4.2), one can find

$$S = \pi r_h^2 , \quad (5.2)$$

which is interesting since it preserves the positivity of the entropic growth.

From the classical expression of heat capacity, one can find the alternative form in terms of the mass of the black hole

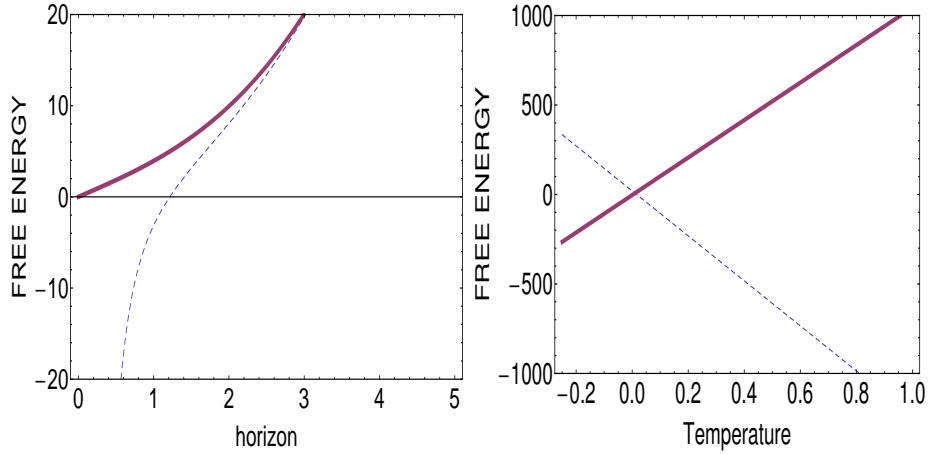
$$C = \left( \frac{\partial M}{\partial T} \right) = \left( \frac{\partial M}{\partial r_h} \right) \left( \frac{\partial r_h}{\partial T} \right) ,$$

and obtain the resulting formula

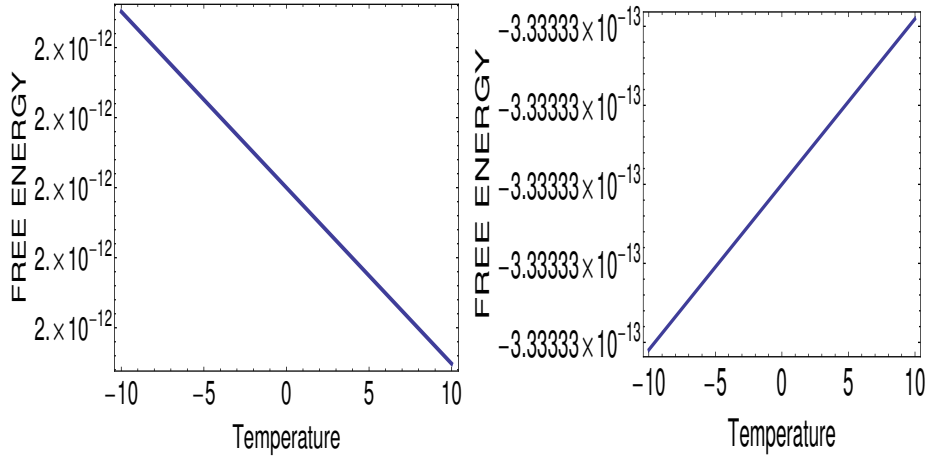
$$C = 2\pi r_h^2 \frac{(1 + 9\alpha_0^2 r^{2(1-n)})}{9\alpha_0^2(1 - 2n)r^{2(1-n)} - 1} . \quad (5.3)$$

In order to analyze the thermal stability, we plot the resulting figures of heat capacity in terms of horizon as shown in Fig.(1).

The positive values of heat capacity indicate a local stability of the system against perturbations. As the plots indicate in the left panel Fig.(1), the initial constraints with  $-1/2 \leq n \leq 3$ , as shown in dashed line, lead to negative values for heat capacity, which means that we find an unstable black hole. Interestingly, a very different situation happens to a more tight range at  $-1/2 \leq n \leq 1.8$ , as indicated by the left panel in Fig.(1) in solid line.



**Figure 2.** The present figures show free energy as a function of the horizon and temperature for the allowed values of  $n$ . In the left panel is shown the behavior for the range with the allowed values  $-1/2 \leq n \leq 1.8$  (solid line) and for the values  $-1/2 \leq n \leq 3$  (dashed line) with free energy as a function of the horizon. In the right panel we compare free energy to temperature and is shown the behavior for the ranges for the allowed values  $-1/2 \leq n \leq 1.8$  (solid line) and  $-1/2 \leq n \leq 3$  (dashed line).



**Figure 3.** The present figures show free energy as a function of temperature  $T$  for very small black holes. In left panel is shown the behavior for the range for the allowed values  $-1/2 \leq n \leq 3$  and in right panel for the case  $-1/2 \leq n \leq 1.8$ .

For small black holes we have an unstable solution. At the point of the horizon around at 0.5 we find the extreme points of heat capacity indicating a phase transition. Accordingly, after that point, the larger is the horizon the more stable is the black hole against perturbations. In the right in Fig.(1), we compare the case  $-1/2 \leq n \leq 3$  (solid line) to the standard Schwarzschild solution (dashed line). It results that the solution in the range  $-1/2 \leq n \leq 3$  is more unstable than the Schwarzschild one.

Another important aspect is to study the global strength of stability which is given by the free energy. When it is related to temperature, it can be a valuable tool to study phase

transition regimes. The Helmholtz free energy is given by

$$F = M - TS ,$$

and with Eqs.(4.2), (5.1) and (5.2), one can obtain

$$F = \frac{1}{2} (r_h + 9\alpha_0^2 r_h^{3-2n}) .$$

The lesser values of the free energy indicate a global stability and the occurrence of Hawking-Page phase transition [40, 41]. We have checked two cases considering the free energy as a function of the horizons and latter as a function of temperature, as shown in the left and right panels in Fig.(2).

As the plots indicate in the left panel in Fig.(2), the initial constraints with  $-1/2 \leq n \leq 3$ , as shown in dashed line, lead to negative values for the free energy and then to unstable black holes as the horizon grows. A narrow window for transitory stability occurs around at  $0.5 \leq r_h \leq 1$ , which is compatible with the left panel in Fig.(1). A similar pattern occurs when considering the temperature as shown in the right panel in Fig.(2) for a fixed radius  $R = 10$ . The transition phase occurs only at the origin with  $T = 0$ , so we have the extremal case. For hot space, we do not have phase transitions due to the fact that the free energy drops to negative values, which leads to several issues [15, 42, 43]. Conversely, we have an unstable solution as the heat capacity indicates.

It is important to point out that in this case the higher is the free energy, the lower is the temperature leading to negative values, which can be discarded if we neglect unstable solutions. The same conclusion we find for very large black holes (with horizon  $r_h \sim 10^{12}$ ). For very small black holes (with  $r_h \sim 10^{-12}$ ), as shown in left panel in Fig.(3), the free-energy is positive and tends to a fixed value of  $T = 10$ . Thus, the system is not thermically stable and transition phases do not occur. The conclusion is that in that range, it does not provide stable small black holes.

In the second case with  $-1/2 \leq n \leq 1.8$ , as shown in solid line in the left panel in Fig.(2), a phase transition occurs around zero free energy, not much different from the region presented in the left panel in Fig.(3). As we consider the temperature, as we go to  $T = 0$  in right panel in Fig.(2), the free energy goes to a maximum at zero which indicates a thermal equilibrium. It means that for  $R = 10$ , and so on, we will not have a phase transition which is compatible with the heat capacity behavior. As a result, a large black hole will not have a phase transition as it grows if the temperature can be kept around zero, which defines an extremal black hole. Differently from the previous one, this case provides negative values for free energy which can give a more global stable situation, but with the price that the temperature must be negative. The same situation we have observed for very large black holes (with  $r_h \sim 10^{12}$ ).

On the other hand, for very small black holes with a radius of the order of  $10^{-2}$ , we have observed that they reach stability next to  $T = 5$ . At below radius of the order of  $10^{-3}$  and so on, they stabilize at  $T = 10$ , and the free-energy continues to drop down. For a horizon of the order of  $10^{-12}$ , free energy is quite small of the order of  $10^{-13}$ , which leads the conclusion that only small black holes can be stable at a certain range of temperature  $0 \leq T \leq 10$ . As an overall conclusion, the best solution occurs at  $-1/2 \leq n \leq 1.8$  predicting large black holes locally stable, but not globally preferred.

## 6 Remarks

In this paper, we have discussed the local and global thermodynamic classical stability properties for a class of static black holes embedded in a five-dimensional bulk space-time. Applying Nash embedding theorem to a static spherical symmetric metric, we have found a modification induced by the extrinsic curvature. Due to the fact that this metric has a restricted embedding in five-dimensional bulk space, we have dealt with the homogeneity of Codazzi equation in Eq.(2.14) using the asymptotic condition on extrinsic curvature based on the smooth principle of Nash theorem. Hence, the metric perturbation and the embedding lead naturally to a higher dimensional structure.

We have shown that an analytical solution for embedded spherically static black holes is obtained. Analyzing the modified line element in Eq.(3.17) we have obtained the related horizons, which we have studied the behavior of a light ray near such horizons. As a result, we have found that the more extrinsic curvature is felt the more is the deformation on the space-time and the deviation is stronger than that one obtained from general relativity.

Concerning the horizons, we have got a set of scalar fields initially constrained by the potentials produced in solar scale. Moreover, the determination of thermal quantities for such black holes, we have found a tight constraint for such set of scalar fields varying at  $-1/2 \leq n \leq 1.8$ . As an overall conclusion, we have obtained a restricted model of stable modified black holes (initially spherically symmetric) as compared to RS models. In this range, we have obtained a local stable behavior for large black holes with a phase transition at  $r_h \sim 0.5$ .

We also have shown that the global stability is not a preferred state possibly influenced by the restricted embedding in five-dimensions, which may seem not suffice for an appropriate description of the phenomena. Accordingly, in order to obtain a more general situation, both global and local stability, an embedding in a more larger bulk seems an unavoidable situation. The entropy is that one expected  $\pi r_h^2$  that grows positively. In RS model solutions for a Schwarzschild back-hole seems to be unstable [44]. In addition, since our analysis is classical, mini black holes are not allowed.

Differently from RS model, we obtained a richer general set of static Black holes, e.g, Bardeen-like black holes and Reissner-Nordström-like black-holes. In this particular case, our tidal charge  $\alpha_0$  is originated from the extrinsic curvature and has a new physics associated to it [1–3], since it has a cosmological magnitude. As future prospects, a more general study of including rotation should be made concerning the canonical and grand canonical ensemble in a higher dimension, which are in due course.

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